Stochastic Calculus - Solution to the Exam

Paris Dauphine University - Master I.E.F. (272)

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$?^{st}$ December 2018.

Solution 1 (4 pts)

a) (2 pts) At date 1, the price of the put option is either $P_1^u = \max\{K - S_1^u; 0\} = 0$ or $P_1^d = \max\{K - S_1^d; 0\} = 3$.

The equivalent martingale measure is

$$q = \frac{1+r-d}{u-d} = \frac{1.02 - \frac{S_1^d}{S_0}}{\frac{S_1^u}{S_0} - \frac{S_1^d}{S_0}} = \frac{1.02 \times S_0 - S_1^d}{S_1^u - S_1^d} = \frac{1.02 \times 20 - 19}{24 - 19} = 0.28$$

The non-arbitrage price of the put option is then

$$P_0 = \frac{qP_1^u + (1-q)\,P_1^d}{1+r} = \frac{0.28 \times 0 + (1-0.28) \times 3}{1.02} \simeq 2.117\,6 \simeq 2.12$$

b) (2 pts) The market price of the option is below its non-arbitrage price. An arbitrage portfolio could then consist in, at time 0 :

- selling the option at its market price (receiving 2.25€) ;

- short-selling $(-\Delta_0)$ units of the risky asset, with $\Delta_0 = \frac{P_1^u - P_1^d}{S_1^u - S_1^d} = \frac{0-3}{24-19} = -\frac{3}{5} = -0.6$. Then receiving $(-\Delta_0) \times S_0 = 0.6 \times 20 = 12$€ ; and

- investing the sum of the two $(2.25 + 12 = 14.25)$ in the money market.

At time 1 :

- if there is an upward move, the put is not exercised ;

we need $\Delta \times S_1^u = 0.6 \times 24 = 14.4$€ to buy and deliver 0.6 shares of the stock.

We receive $14.25 \times 1.02 = 14.535$€ from the money market. So, the granted profit is :

$$14.535 - 14.4 = 0.135$$

- if there is a downward move, the put is exercised, and we give 3€ to the holder of the put ;

we need $\Delta \times S_1^d = 0.6 \times 19 = 11.4$€ to buy and deliver 0.6 shares of the stock.

We receive $14.25 \times 1.02 = 14.535$€ from the money market. So, the granted profit is :

$$14.535 - 11.4 - 3 = 0.135$$

Hence, in both cases the granted profit is 0.135€.

Solution 2 (7 pts)

a) (1 pt) The price of the stock at time t writes as

$$\begin{aligned}
S_1^u &= 1.04 \times S_0 = 1.04 \times 100 = 104 \\
S_1^d &= 0.95 \times S_0 = 0.95 \times 100 = 95 \\
S_2^{u^2} &= 1.04 \times S_1^u = 1.04 \times 104 \simeq 108.16 \\
S_2^{du} &= 1.04 \times S_1^d = 1.04 \times 95 \simeq 98.8 \\
S_2^{d^2} &= 0.95 \times S_1^d = 0.95 \times 95 \simeq 90.25
\end{aligned}$$

So, the binomial tree that depicts the evolution of the stock price through time t, with $t \in \{0,1,2\}$ is

Figure 1

b) (3 pts) The no-arbitrage price of the first derivative at time 2 writes as

$$\begin{aligned}
E_2 &= \max\{S_2 - (1.5S_2 - 60)\,;0\} \\
&= \max\{-0.5 \times S_2 + 60; 0\}
\end{aligned}$$

where S_2 denotes the price of the underlying asset at time 2. According to the different possible scenario with respect to S_2, we have :

$$\begin{aligned}
E_2^{u^2} &= \max\{-0.5 \times S_2^{u^2} + 60; 0\} \\
&= \max\{-0.5 \times 108.16 + 60; 0\} \simeq 5.92 \\
E_2^{du} &= \max\{-0.5 \times S_2^{du} + 60; 0\} \\
&= \max\{-0.5 \times 98.8 + 60; 0\} \simeq 10.6 \\
E_2^{d^2} &= \max\{-0.5 \times S_2^{d^2} + 60; 0\} \\
&= \max\{-0.5 \times 90.25 + 60; 0\} \simeq 14.88
\end{aligned}$$

The equivalent martingale measure writes as

$$q = \frac{e^{0.02} - 0.95}{1.04 - 0.95} \simeq 0.78$$

So, the no-arbitrage price of the first derivative at time $t \in \{0,1\}$ writes as

$$\begin{aligned}
E_1^u &= \frac{qE_2^{u^2} + (1-q)\,E_2^{du}}{e^{0.02}} \simeq \frac{0.78 \times 5.92 + 0.22 \times 10.60}{e^{0.02}} \simeq 6.81 \\
E_1^d &= \frac{qE_2^{du} + (1-q)\,E_2^{d^2}}{e^{0.02}} \simeq \frac{0.78 \times 10.60 + 0.22 \times 14.88}{e^{0.02}} \simeq 11.31 \\
E_0 &= \frac{qE_1^u + (1-q)\,E_1^d}{e^{0.02}} \simeq \frac{0.78 \times 6.81 + 0.22 \times 11.31}{e^{0.02}} \simeq 7.65
\end{aligned}$$

Hence, the binomial tree that depicts the evolution of the first derivative price through time t, with $t \in \{0,1,2\}$ is

Figure 2

c) (3 pts) The no-arbitrage price of the second derivative at time 2 writes as

$$A_2^x = E_2^x \text{ for any } x \in \{u^2, ud, d^2\}.$$

The no-arbitrage price of the second derivative at time 1 writes as

$$A_1 = \max\{S_1 - (1.5S_1 - 60)\,; E_1\} = \max\{-0.5 \times S_1 + 60; E_1\}$$

The no-arbitrage price of the second derivative at time 0 writes as

$$A_0 = \max\{-0.5 \times S_0 + 60; \frac{qA_1^u + (1-q)\,A_1^d}{e^r}\}.$$

According to the different possible scenario with respect to the underlying asset, we have :

$$\begin{aligned}
A_1^u &= \max\{-0.5 \times S_1^u + 60; E_1^u\} = \max\{-0.5 \times 104 + 60; 6.81\} \\
&\simeq \max\{8.00; 6.81\} = 8.00 \\
A_1^d &= \max\{-0.5 \times S_1^d + 60; E_1^d\} = \max\{-0.5 \times 95 + 60; 11.31\} \\
&\simeq \max\{12.5; 11.31\} = 12.5 \\
A_0 &= \max\{-0.5 \times S_0 + 60; \frac{qA_1^u + (1-q)\,A_1^d}{e^r}\} \\
&= \max\{-0.5 \times 100 + 60; \frac{0.78 \times 8.00 + 0.22 \times 12.50}{e^{0.02}}\} \\
&\simeq \max\{10.00; 8.81\} = 10.00
\end{aligned}$$

Hence, the binomial tree that depicts the evolution of the second derivative price through time t, with $t \in \{0, 1, 2\}$ is

Figure 3

Solution to the Problem (9 pts)

(a) (1 pt) The no-arbitrage price of the option at maturity writes as

$$V_T^K = 1_{\{S_T > K\}}$$

(b) (4 pts) In Black-Scholes world the underlying asset price at maturity T (and any subsequent time t, replacing T with t) is

$$S_T = S_0 R(T)$$

where the log return of the underlying asset price is normally distributed under the equivalent martingale measure $\mathbb{Q}$, with mean $\left(r - \frac{1}{2}\sigma^2\right)T$ and variance $\sigma^2 T$:

$$\ln R(T) \overset{\mathbb{Q}}{\sim} N\left(\left(r - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$$

where σ (resp. r) denotes the volatility of the underlying asset (resp. the risk-free interest rate).

In this setup, the no-arbitrage price at date of issuance of the option must satisfy

$$V_0^K = e^{-rT}\mathbb{E}^{\mathbb{Q}}[V_T^K].$$

We have $\mathbb{E}^{\mathbb{Q}}[V_T^K] = \mathbb{Q}[S_T > K]$. Let us compute $\mathbb{Q}[S_T > K]$. From $S_T = S_0 R(T)$ we have

$$\ln S_T = \ln S_0 + \ln R(T).$$

From

$$\ln R(T) \overset{\mathbb{Q}}{\sim} N\left(\left(r-\frac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$$

we have

$$\ln S_T \overset{\mathbb{Q}}{\sim} N\left(\ln S_0 + \left(r-\frac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$$

So,

$$\frac{\ln S_T - \mathbb{E}[\ln S_T]}{\sqrt{\mathbb{V}[\ln S_T]}} \overset{\mathbb{Q}}{\sim} N(0,1)$$

That is

$$\frac{\ln S_T - \left(\ln S_0 + \left(r-\frac{1}{2}\sigma^2\right)T\right)}{\sqrt{\sigma^2 T}} \overset{\mathbb{Q}}{\sim} N(0,1).$$

Now, let us denote $\mathcal{N}(x) := \mathbb{P}[X \le x]$ when $X \overset{\mathbb{P}}{\sim} N(0,1)$. So we have

$$\mathbb{P}[X > x] = 1 - \mathcal{N}(x) = \mathbb{P}[X < -x] = \mathcal{N}(-x)$$

Using that $S_T > K$ is equivalent to

$$\frac{\ln S_T - \left(\ln S_0 + \left(r-\frac{1}{2}\sigma^2\right)T\right)}{\sqrt{\sigma^2 T}} > \frac{\ln K - \left(\ln S_0 + \left(r-\frac{1}{2}\sigma^2\right)T\right)}{\sqrt{\sigma^2 T}}$$

We obtain

$$\mathbb{Q}[S_T > K] = \mathcal{N}\left(-\frac{\ln K - \left(\ln S_0 + \left(r-\frac{1}{2}\sigma^2\right)T\right)}{\sqrt{\sigma^2 T}}\right)$$

where $\mathcal{N}(\cdot)$ denotes the cumulative probability distribution function for a standardized normal distribution. That is

$$\mathbb{Q}[S_T > K] = \mathcal{N}\left(\frac{\ln \frac{S_0}{K} + \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

Hence,

$$V_0^K = e^{-rT}\mathcal{N}\left(\frac{\ln \frac{S_0}{K} + \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

(c) (1 pt) The no-arbitrage price of the option at any date $t \in (0,T)$, denoted as V_t^K, satisfies then

$$V_t^K = e^{-r(T-t)}\mathcal{N}\left(\frac{\ln \frac{S_0}{K} + \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right)$$

(d) (3 pt) At maturity, the option pays $W_T^{K_1,K_2} = \left(1_{\{S_T>K_1\}} + 1_{\{S_T>K_2\}}\right)€$. Observes that

$$W_T^{K_1,K_2} = V_T^{K_1} + V_T^{K_2}$$

so, at any date $t \in [0,T)$ we have

$$\begin{aligned}
W_t^{K_1,K_2} &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[V_t^{K_1} + V_t^{K_2}] \\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}[V_t^{K_1}] + e^{-rT}\mathbb{E}^{\mathbb{Q}}[V_t^{K_2}] \\
&= V_t^{K_1} + V_t^{K_2} \\
&= e^{-r(T-t)}\left(\mathcal{N}\left(\frac{\ln \frac{S_0}{K_1} + \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right) + \mathcal{N}\left(\frac{\ln \frac{S_0}{K_2} + \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right)\right)
\end{aligned}$$

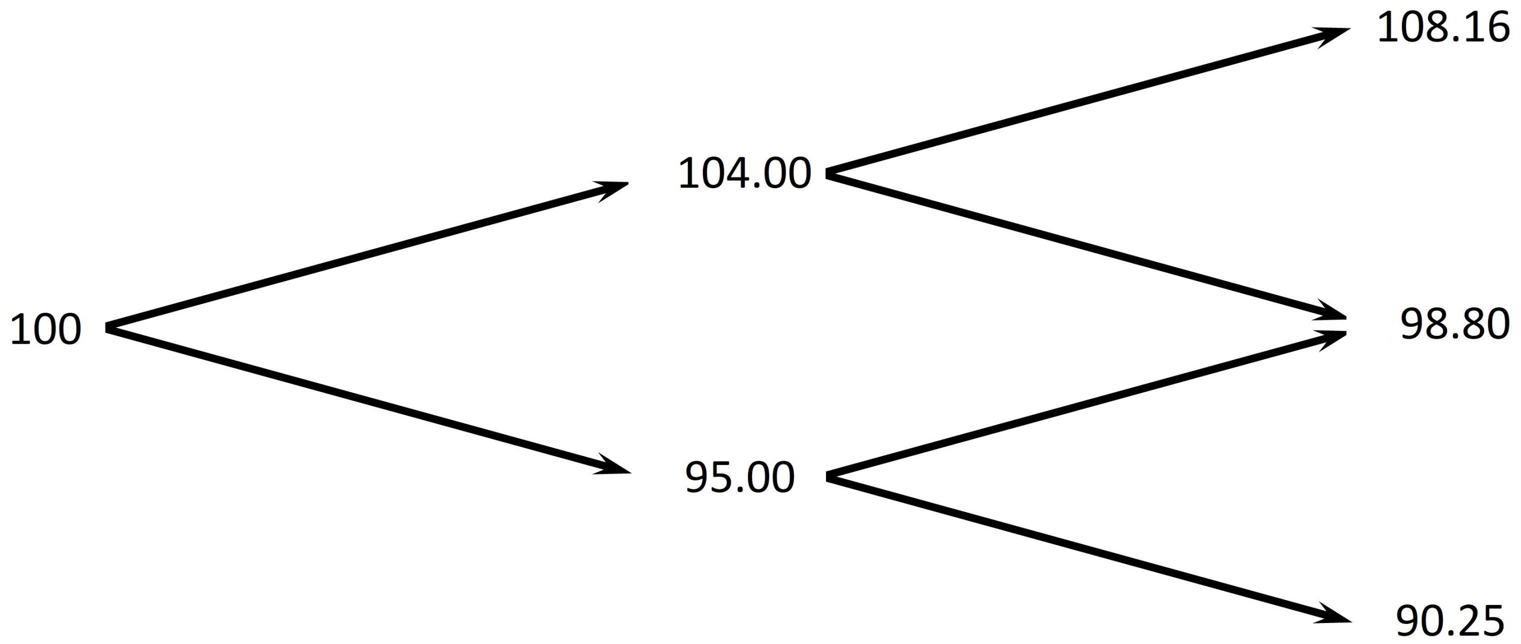

Figure 1: Evolution of the stock price through time

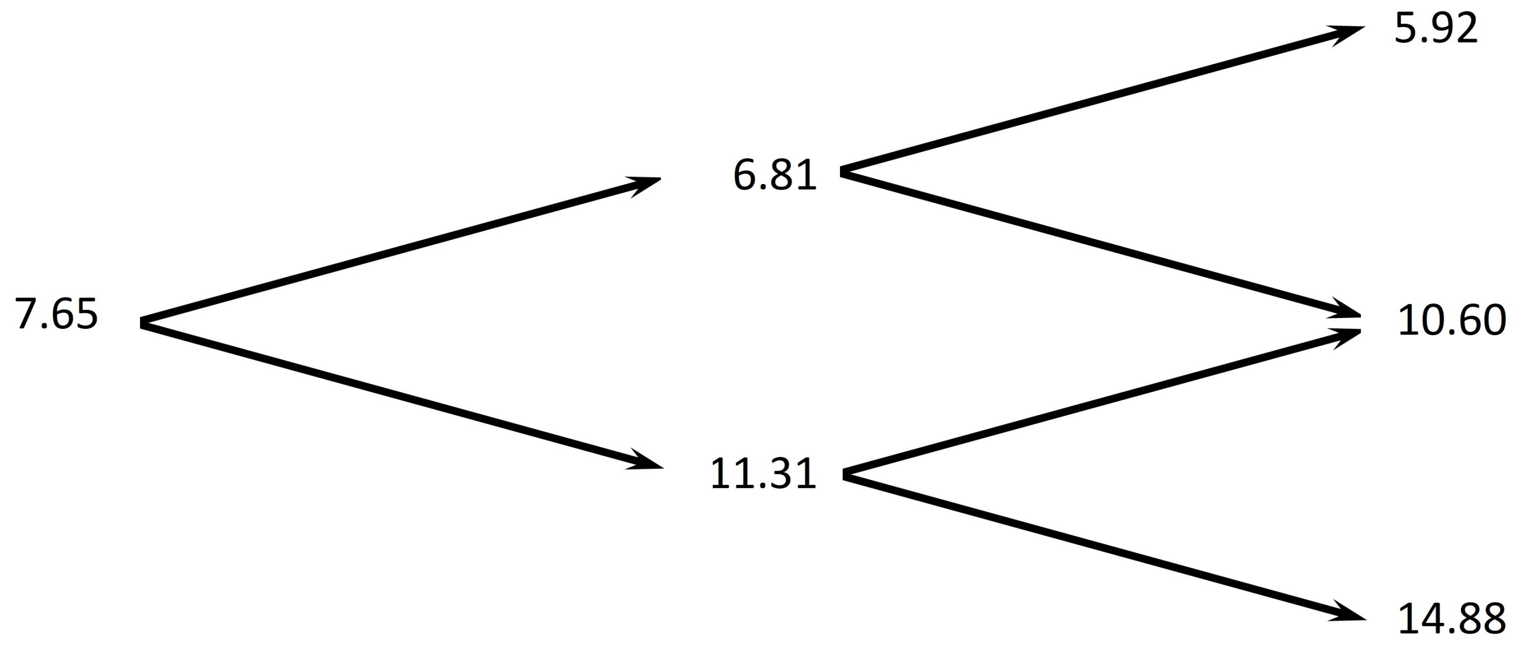

Figure 2: Evolution of the first derivative price through time

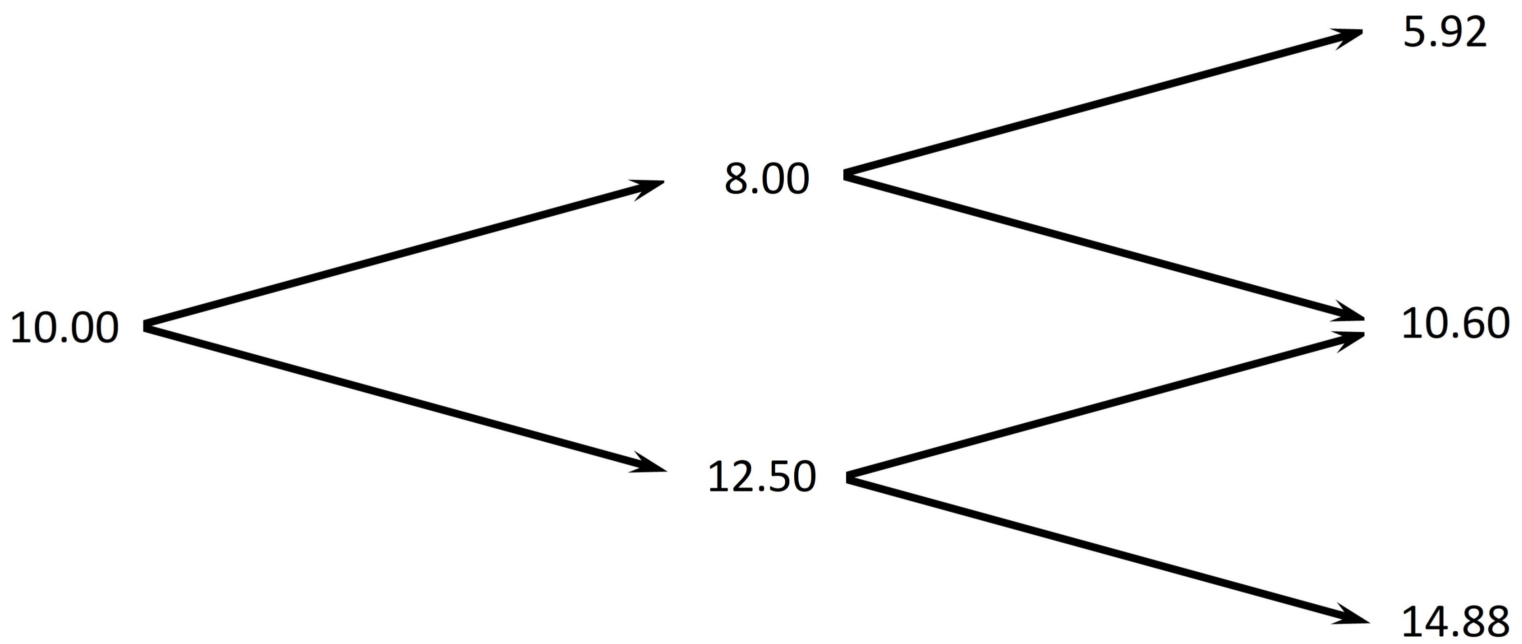

Figure 3: Evolution of the second derivative price through time