Stochastic Calculus - Exam

Paris Dauphine University - Master I.E.F. (272)

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No document allowed. Calculator allowed.
Answers can be formulated in English or French.

Exercise 1 ***(4 pts).*** *There are two periods, $t \in \{0, 1\}$. There are two assets. One non-risky asset (money that can be borrowed or lend) that returns $r = 2\%$ with discrete compounding at time 1. And one risky asset which is a stock of price $S_0 = 20$ at time 0. At date 1, there is either an upward or a downward move. The price of the stock is then either $S_1^u = 24$ or $S_1^d = 19$.*

Suppose the market price of an European put option on the stock with strike 22€ at time 0 is 2.25€.

a) (2 pts) *What should be the non-arbitrage price of the put option at date 0 ?*

b) (2 pts) *Construct an arbitrage portfolio that uses one unit of the put option.*

Exercise 2 ***(7 pts)*** *Consider a stock whose price starts at $S_0 = 100$€ and evolves according to a two-steps binomial tree where each upward (resp. downward) move increases (resp. decreases) the value by 4% (resp. by 5%). The risk-free interest rate is 2% and is continuously compounded. At date $t = 0$, a financial institution issues two derivatives that each matures at time $t = 2$.*

According to the underlying contracts, the buyer of the first (resp. second) derivative has the right to buy one unit of the stock at time $t = 2$ (resp. at any time $t \in \{0, 1, 2\}$), for a price $1.5S_2 - 60$ (resp. $1.5S_t - 60$).

a) (1 pt) *Draw the binomial tree that depicts the evolution of the stock price through time t, with $t \in \{0, 1, 2\}$.*

b) (3 pts) *Draw the binomial tree that depicts the evolution of the first derivative no-arbitrage price, denoted as E_t, through time t, with $t \in \{0, 1, 2\}$.*

c) (3 pts) *Draw the binomial tree that depicts the evolution of the second derivative no-arbitrage price, denoted as A_t, through time t, with $t \in \{0, 1, 2\}$.*

Problem 3 ***(9 pts)*** *Consider an option in Black-Scholes world that pays you 1€ at maturity T if the price of the underlying asset S_T is higher than a given "strike price" K, and pays you 0€ otherwise.*

(a) (1 pt) *Give the no-arbitrage price of the option at maturity, denoted as V_T^K.*

(b) (4 pt) *Give the no-arbitrage price of the option at the date of issuance, denoted as V_0^K.*

(c) (1 pt) *Deduce from the previous answer the no-arbitrage price of the option at any date $t \in (0, T)$, denoted as V_t^K.*

(d) (3 pt) *Consider now an option in Black-Scholes world that pays you 1€ at maturity T if $K_1 < S_T \leq K_2$, pays you 2€ if $S_T > K_2$, and pays you 0€ otherwise, where K_1 and K_2 are two given "strike prices", with $0 \leq K_1 < K_2$. Give the no-arbitrage price of the option at maturity, denoted as $W_T^{K_1,K_2}$, and the no-arbitrage price at any date $t \in (0, T)$, denoted as $W_t^{K_1,K_2}$.*